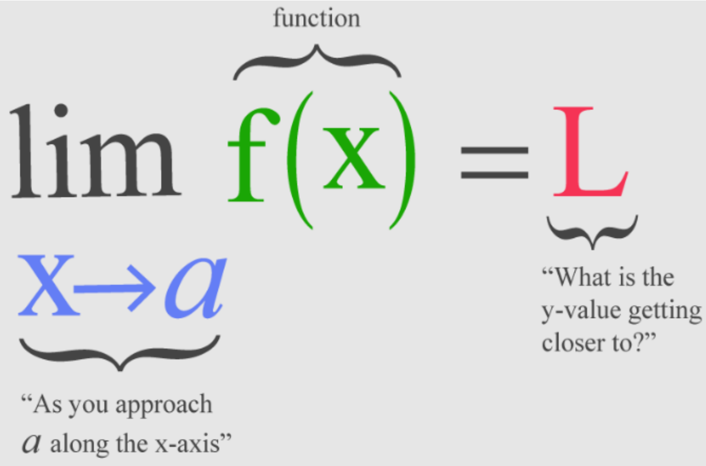
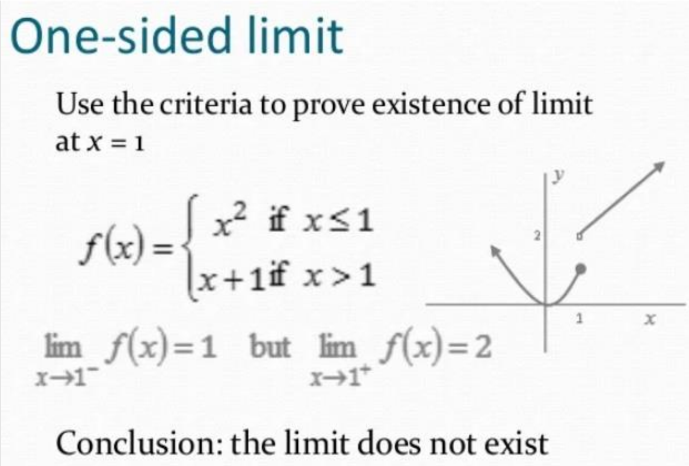
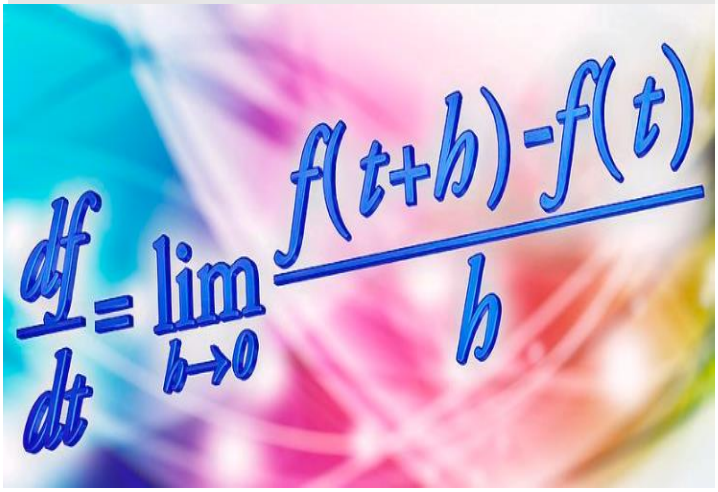
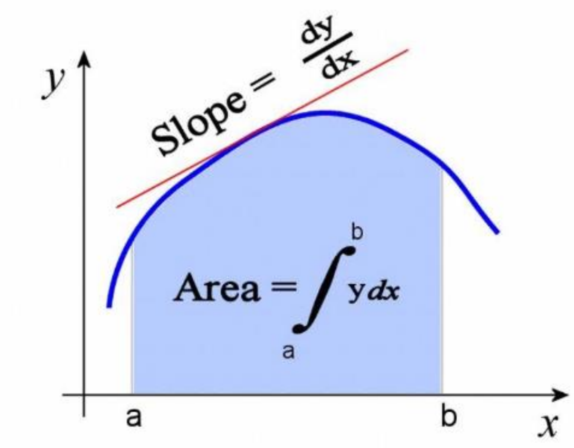
Military Technological College









GFP- Pure Mathematics

MODULE CODE: MTCG1018

WORKBOOK- 3

Learning Outcomes – On successful completion of this module, students should be able to:	
1.	Demonstrate understanding of the definition of a function and its graph.
2.	Define and manipulate exponential and logarithmic functions and solve problems arising from real life applications.
3.	Understand the inverse relationship between exponents and logarithms functions and use this relationship to solve related problems.
4.	Understand basic concepts of descriptive statistics, mean, median, mode and summarize data into tables and simple graphs (bar charts, histogram, and pie chart).
5.	Understand basic probability concepts and compute the probability of simple events using tree diagrams and formulas for permutations and combinations.
6.	Define and evaluate limit of a function as well as test continuity of a function.
7.	Determine the surface areas, the volumes and capacities of common shapes and 3-dimesions figures (square, rectangle, parallelogram, trapezium, cuboid, cone, pyramid and prisms).
8.	Find the derivatives of standard and composite functions using standard rules of differentiation.
9.	Use the law of sines and cosines to solve a triangle and real-life problems.

MILITARY TECHNOLOGICAL COLLEGE

Delivery Plan - Year 2024-25
[Term 2]

Title / Module Code / Programme	Pure Mathematics /MTCG1018/Foundation Programme Department (FPD)	Module Coordinator	Mr. Knowledge Simango
Lecturers	TBA	Resources & Reference books	Moodle & Workbook
Duration & Contact Hours	Term 2: 4 hrs x 11 weeks = 44 hours		

Week No.	TOPICS	Hours	Learning Outcome No.
1	**Introduction** **1. Law of sines and cosines to solve a triangle** 1.1 Law of sines 1.2 Law of cosines **2. Perimeter, Area and Volume** 2.1 Perimeter and area	4	7, 9
2	2.2 Volume and surface area **3. Statistics** 3.1 Basic concepts of descriptive statistics 3.2 Types of Data	4	4, 7
3	3.3 Summarizing and presenting data. 3.4 Measures of Central Tendency 3.5 Measures of Dispersion **Revision for Continuous Assessment-1**	4	4
	Continuous Assessment-1 (Chapter 1 and 2)		**7 and 9**
4	**4. Probability** 4.1 Basic Concepts 4.2 Probability 4.3 Rules of Probability	4	5
5	**5. Functions and graphs** 5.1 Domain, range and function 5.2 Types of functions 5.3 Inverse function	4	1
6	5.4 Operations of functions 5.5 Composite function **6. Exponential functions** 6.1 Exponential equations	4	1
7	6.2 Exponential function and graphs 6.3 Application in real life **Revision for Continuous Assessment-2**	4	2
	Continuous Assessment-2 (Chapter 3, 4 and 5)		**1, 4 and 5**

8	**7. logarithmic functions** 7.1 Logarithm Definition and Properties 7.2 Logarithmic function and graph 7.3 Exponential and logarithmic equations **8. Limits** 8.1 Basic Concepts of Limit	**4**	**2, 3, 6**
9	8.2 Methods of finding limits 8.3 Limits at Infinity 8.4 Continuity of a Function **9. Differentiation** 9.1 The Gradient of a Curve	**4**	**6, 8**
10	9.2 Differentiation from the First Principles 9.3 Methods of Differentiation	**4**	**8**
11	9.4 Applications of Derivatives	**4**	**8**
	Revision for Final Exam,		**2, 3, 6 & 8**
12/13	**FINAL EXAM (Unit-6 to Unit-9)**		**2, 3, 6 & 8**
	Total hours	**44**	

Indicative Reading	
Title/Edition/Author	**ISBN**
College Algebra with Trigonometry-7th Edition by K Raymond A., Ziegler Michael R., Byleen	ISBN-13: 978-0072368697 ISBN-10: 0072368691
College Algebra and Trigonometry-5th Edition by Margaret L. Lial, John Hornsby, David I. Schneider and Callie Daniels	ISBN-13: 978-0321671783 ISBN-10: 0321671783
Bird's Basic Engineering Mathematics- 8th Edition by John Bird	ISBN-13: 978-0367643706 ISBN-10: 0367643707
Engineering Mathematics- 8th Edition by K.A. Stroud and Dexter Booth	ISBN-13: 978-1352010275 ISBN-10: 1352010275

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Dr. T Raja Rani
DHOD FPD(CMP)

MQM Salim Al Shibli
Head FPD

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(UNIT-6) EXPONENTIAL FUNCTIONS

6.1 EXPONENTIAL EQUATIONS

An **exponential equation is** an equation involving expressions having exponents that are unknown. The variable is on the exponent of a term in the equation. The laws of exponents or indices are useful in solving an exponential equation.

Laws of exponents or indices:

1) $a^x a^y = a^{x+y}$
2) $\left(a^x\right)^y = a^{xy}$
3) $(ab)^x = a^x b^x$
4) $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
5) $\dfrac{a^x}{a^y} = a^{x-y}$

Where ***a* and *b*** are positive, and ***x and y*** are real numbers.

Note:

1) $a^x = a^y$ if and only if $x = y$

2) $a^x = b^x$ if and only if $a = b$

Example 1: Solve $4^{x-3} = 8$ for x

Solution: $4^{x-3} = 8 = 2^3$

$$(2^2)^{x-3} = 2^3$$

$$2(x-3) = 3$$

$$2x - 6 = 3$$

$$x = \frac{9}{2}$$

Class Activity

1) Solve $27^{x+1} = 9$ for x

2) Solve $2^{2x+1} = 4$ for x

6.2 EXPONENTIAL FUNCTIONS AND GRAPHS

Exponential Function

The equation

$f(x) = a^x$ *where* $a > 0,\ a \neq 1$

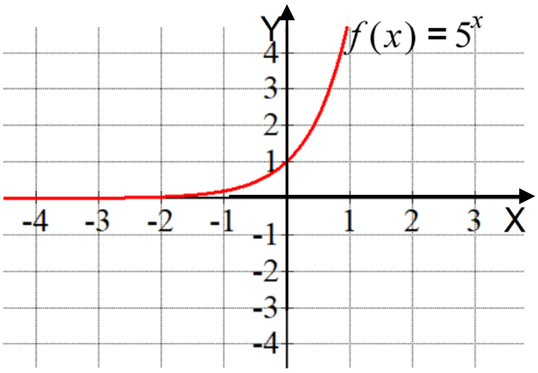
is called an **exponential function.** The constant a is called the **base** and x is called the **exponent or power.**

Examples: $y = 2^x$, $y = 0.5^{2x}$, $y = \left(\frac{1}{3}\right)^x$

Basic exponential graphs

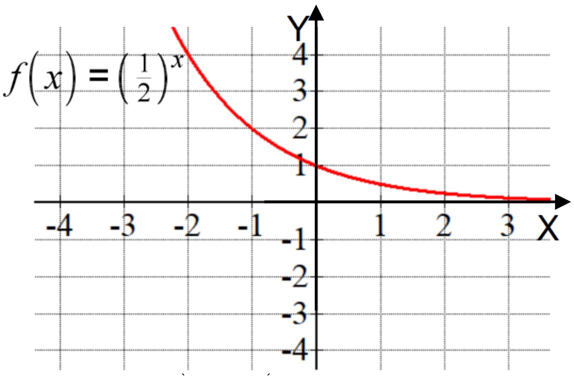
There are two cases in exponential functions.

Case 1: $f(x) = a^x$ *where* $a > 1$, here $a = 5$



Case 2: $f(x) = a^x$ *where* $0 < a < 1$,

here $a = \frac{1}{2}$



Basic properties of exponential graphs:

1) The domain of f is the set of all real numbers $(-\infty, \infty)$
2) The range of f is the set of all positive real numbers $(0, \infty)$.
3) All graphs pass through the point (0, 1).
4) All graphs are continuous that is, there are no holes or jumps.
5) The X-axis is a horizontal asymptote, that is, there is no intercept on X-axis.
6) If $a > 1$, then a^x increases as x increases.
7) If $0 < a < 1$, then a^x decreases as x increases.
8) The function is one to one.

Exponential function with base e

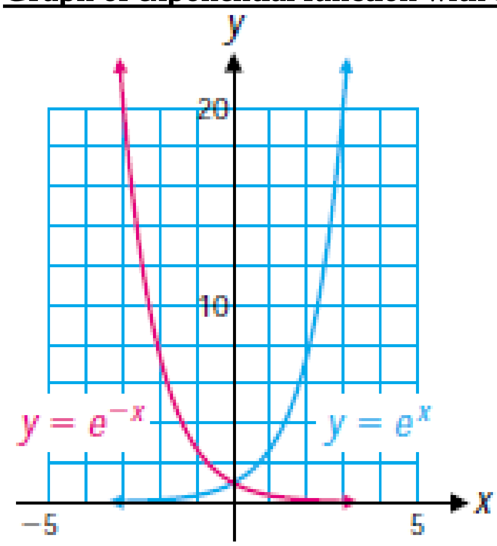
The equation $f(x) = e^x$,

where x is a real number, is called an **exponential function with base *e*.**

Note: *e* = 2.718 281 828 459 …

The constant e turns out to be an ideal base for an exponential function because in calculus and higher mathematics many operations take on their simplest form using this base.

Graph of exponential function with base *e*



Graphing of exponential functions

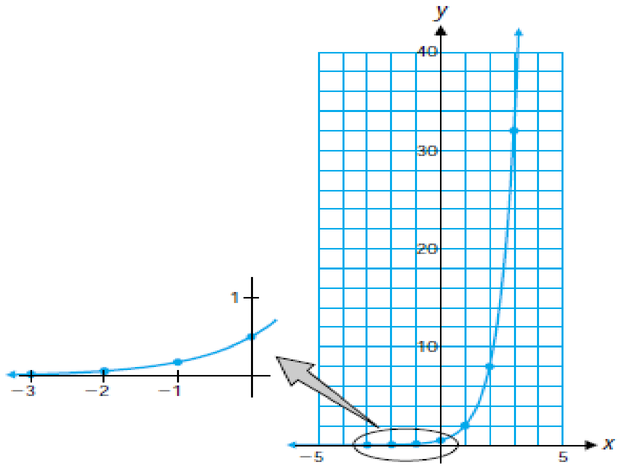
Example 1: Use integer values of x from -3 to 3 to construct a table of values for

$$y = \frac{1}{2}(4^x)$$

Method : Use a calculator to create the table of values shown below

x	y
−3	0.01
−2	0.03
−1	0.13
0	0.50
1	2.00
2	8.00
3	32.00

Then plot the points and join these points with a smooth curve



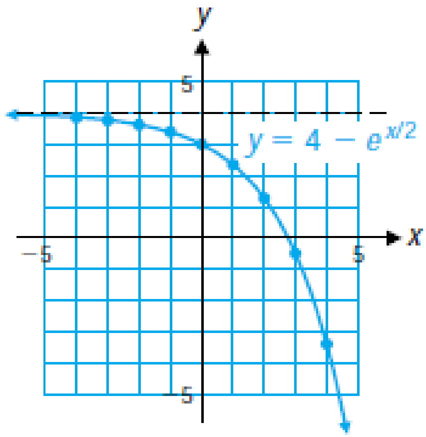
Example 2: Use integer values of x from -4 to 4 to construct a table of values for

$$y = 4 - e^{\frac{x}{2}}$$

Method: Use a calculator to create the table of values shown below

x	y
−4	3.86
−3	3.78
−2	3.63
−1	3.39
0	3
1	2.35
2	1.28
3	−0.48
4	−3.39

Then plot the points and join these points with a smooth curve



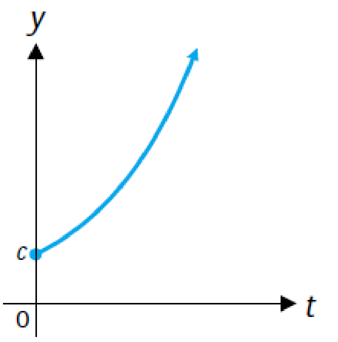
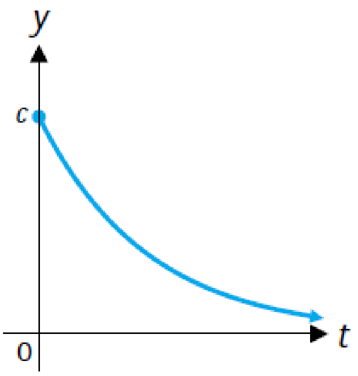
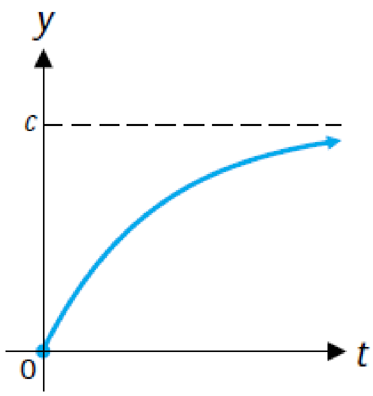
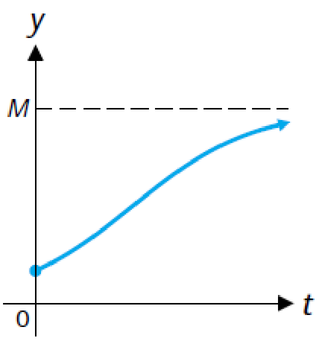
Class Activity

1) Use integer values of x from -3 to 3 to construct a table of values for $y = \frac{1}{2}(4^{-x})$, and then graph this function.

2) Use integer values of x from -4 to 4 to construct a table of values for $y = 2e^{\frac{x}{2}} - 5$ and then graph this function.

6.3 APPLICATIONS IN REAL LIFE

Table-Exponential growth and decay

Description	Equation	Graph	Uses
Unlimited growth	$y = ce^{kt}$ $c, k > 0$		Short-Term population growth (people, bacteria, etc.) growth of money at continuous compound interest
Exponential decay	$y = ce^{-kt}$ $c, k > 0$		Radioactive decay, light absorption in water, glass, etc. atmospheric pressure, electric circuits
Limited growth	$y = c\left(1 - e^{-kt}\right)$ $c, k > 0$		Sales fads, company growth, electric circuits
Logistic growth	$y = \frac{M}{1 + ce^{-kt}}$ $c, k, M > 0$		Long-term population growth, epidemics, sales of new products, company growth

More applications of exponential function

Population growth and compound interest are examples of exponential growth, while radioactive decay is an example of negative exponential growth.

Example 1: Mexico has a population of around 100 million people, and it is estimated that the population will double in 21 years. If population growth continues at the same rate and model of population growth is given by : $P = P_0 2^{\frac{t}{d}}$

Where, P = population at time t
P_0 = population at time $t = 0$
d = doubling time
. What will be the population?

i) 15 years from now?
ii) 30 years from now?

Calculate the answers up to 3 significant digits.

Solution: We use the doubling time growth model: $P = P_0 2^{\frac{t}{d}}$

Substituting $P_0 = 100$ and $d = 21$, we get

$$P = 100\left(2^{\frac{t}{21}}\right)$$

i) When t = 15 years ,

$$P = 100\left(2^{\frac{15}{21}}\right)$$

$$P \approx 164 \text{ million people}$$

ii) When t = 30 years ,

$$P = 100\left(2^{\frac{30}{21}}\right)$$

$$P \approx 269 \text{ million people}$$

Example 2: The rate of decay of radioactive isotope gallium 67 (^{67}Ga), used in the diagnosis of malignant tumors, is modelled as

$$A == A_0 2^{-\frac{t}{h}}$$

where A = amount at time t , A_0 = amount at time $t = 0$ and h = half-life.

If we start with 100 milligrams of the isotope and it has a biological half- life of 46.5 hours, how many milligrams will be left after

i) 24 hours?

ii) 1 week?

Calculate the answers up to 3 significant digits.

Solution: we use the half decay model:

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}} = A_0 2^{-\frac{t}{h}}$$

Substituting $A_0 = 100$ and $h = 46.5$, we get

$$A = 100\left(2^{-\frac{t}{46.5}}\right)$$

i) When t = 24 hours,

$$A = 100\left(2^{-\frac{24}{46.5}}\right) \approx 69.9 \text{ millgrams}$$

ii) When t = 1 week = 168 hours,

$$A = 100\left(2^{-\frac{168}{46.5}}\right) \approx 8.17 \text{ millgrams}$$

Example 3: If a principal P is invested at an annual rate r compounded n times a year, then the amount A at the end of the t years is given by $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

Suppose 1000 RO is deposited in the account paying 4% interest per year compounded quarterly (four times per year).

i) Find the amount in the account after 10 years with no withdrawals.
ii) How much interest is earned over the 10 year period?

Compute the answer to the nearest baiza.

Solution: i) Compound interest formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Here P = 1000, r = 4% =0.04, n = 4 and

t = 10.

$$A = 1000\left(1+\frac{0.04}{4}\right)^{4\times10}$$

$$A = 1000(1+0.01)^{40}$$

A = 1488.86 RO (rounded to nearest baiza)

Thus 1488.86 RO is in account after 10 years.

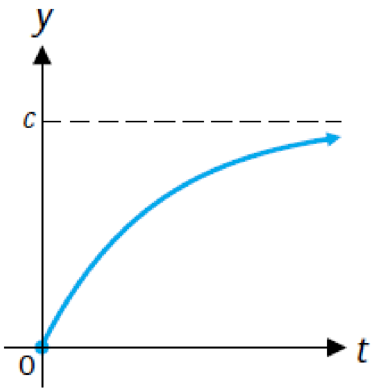
ii) The interest earned for that period is

1488.86 RO – 1000 RO = 488.864 RO

Class Activity

I) Circle the correct answer in the following questions.

(1) The following graph describes …………



(a) Unlimited growth

(b) Limited growth

(c) Exponential decay

II) Show your solution step by step in the following questions.

1) Over short period of times the doubling time growth model is often used to model population growth:

$$P = P_0 2^{\frac{t}{d}}$$

Where, P = population at time t

P_0 = population at time t = 0

d = doubling time

In a particular laboratory, the doubling time for bacterium *Escherichia coli* (*E. Coli*), which is found naturally in the intestines of many mammals, is found to be 25 minutes. If the experiment starts with a population of 1,000 *E. coli* and there is no change in the doubling time, how many bacteria will be present after:

i) 10 minutes?

ii) 5 hours?

Calculate the answers up to 3 significant digits.

Solution:

2) The rate of decay of radioactive gold 198 (^{198}Au), used in imaging the structure of the liver, is modelled as $A == A_0 2^{-\frac{t}{h}}$

where, A = amount at time t , A_0 = amount at time t = 0 and h = half-life.

If we start with 50 milligrams of the isotope and it has a biological half- life of 2.67 days, how many milligrams will be left after:

i) Half day?
ii) 1 week?

Calculate the answers up to 3 significant digits.

Solution:

3) If a principal P is invested at an annual rate r compounded n times a year, then the amount A at the end of the t years is given by $A = P\left(1+\frac{r}{n}\right)^{nt}$.

Suppose 8000 RO is deposited in the account paying 6% interest per year compounded half yearly. Find the amount in the account after 5 years with no withdrawals.

Solution:

WORKSHEET 6

Show your solution step by step in the following questions.

1) Simplify the following:

i) $3^{5x+1}3^{3-2x}$

ii) $e^{x}(e^{-x}+1)-e^{-x}(e^{x}+1)$

2) Solve the following for *x*

i) $2^{5x+1} = 2^{5+2x}$

ii) $8^{2x+1} = 32$

iii) $3^{3x+2} = \frac{1}{81}$

iv) $10^{x^2+2} = 10^{2x+2}$

3) Graph $y = -e^{x}$; $[-3,3]$

4) Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as modeled by $N = N_0 e^{1.386t}$

Where N is the number of bacteria present after *t* hours and N_0 is the number of bacteria present at t =0. If we start with 1 bacterium, how many bacteria will be present in

i) 5 hours?
ii) 12 hours?

Compute the answers to 3 significant digits.

5) If a principal P is invested at an annual rate r compounded n times a year, then the amount A at the end of the t years is given by $A = P\left(1+\frac{r}{n}\right)^{nt}$.

Suppose 5000 RO is deposited in the account paying 9% interest per year compounded daily (365 days).

i) Find the amount in the account after 5 years with no withdrawals.
ii) How much interest is earned over the 5 year period.

Compute the answer to the nearest Baiza

(UNIT-7) LOGARITHM FUNCTIONS

7.1 INTER-CONVERSION OF EXPONENTIAL AND LOGARITHM FUNCTIONS

Definition: Logarithm of a Number

The **logarithm** of a number is the exponent to which the base must be raised to obtain that number.

In general, $\log_a x = n$ implies that $a^n = x$.

and conversely, if $x = a^n$, then $\log_a x = n$ where, $a > 0$, $a \neq 1$, $and\ x > 0$.

$a^n = x$ is the exponential form and $\log_a x = n$ is the logarithmic form.

$2^3 = 8 \longrightarrow \text{Log}_2 8 = 3$

$10^2 = 100 \longrightarrow \text{Log}_{10} 100 = 2$

$10^3 = 1000 \longrightarrow \text{Log}_{10} 1000 = 3$

Class Activity 1

1) **Write each of the following in logarithmic form:**

(i) $2^4 = 16$

(ii) $3^3 = 27$

(iii) $5^3 = 125$

(iv) $3 = \sqrt{9}$

(v) $\frac{1}{5} = 5^{-1}$

2) **Write each of the following logarithms in exponential form:**

(i) $\log_2 16 = 4$

(ii) $\log_4 64 = 3$

(iii) $\log_{10} 1000000 = 6$

(iv) $\log_{25} 5 = \frac{1}{2}$

(v) $\log_2 \frac{1}{4} = -2$

Properties of Logarithms

If a, x and y are positive real numbers, $a \neq 1$ and b is a real number then:

1) $\log_a 1 = 0$

Since $a^0 = 1$, then, $\log_a 1 = 0$

Example : $\log_2 (1) = 0$ and $\log_{25} (1) = 0$, etc.

2) $\log_a a = 1$

Since $a^1 = a$, then, $\log_a a = 1$

Example : $\log_2 2 = 1$ and $\log_{20} 20 = 1$

3) $\log_a xy = \log_a x + \log_a y$

Examples: a) $\log_2(8\times 4) = \log_2 8 + \log_2 4$

b) $\log_3 12 = \log_3(3\times 4) = \log_3 3 + \log_3 4$

4) $\log_a \frac{x}{y} = \log_a x - \log_a y$

Examples: a) $log_2 \frac{100}{3} = log_2 100 - log_2 3$

b) $log_{10} \frac{10000}{10} = log_{10} 10000 - log_{10} 10$

$= 4 - 1 = 3$

5) $log_a x^b = b\ log_a x$

Example 1: $log_{10} 10000 = log_{10} 10^4 = 4\ log_{10} 10 = 4$

Example 2: $log_2\ (\sqrt[3]{5}) = log_2\ \left(5^{\frac{1}{3}}\right)$

$= \frac{1}{3}\ log_2\ (5)$

Therefore, $log_2\ (\sqrt[3]{5}) = \frac{log_2\ 5}{3}$

The above rules are same for all positive bases. The most common bases are the base 10 and the base e. Logarithms with a base 10 are called **common logarithms**, and logarithms with a base e are **natural logarithms**. On your calculator, the base 10 logarithm is noted by **log**, and the base e logarithm is noted by **ln**.

Note: When the base is 10, we do not need to state it.

Class Activity 2

1) Find the values of the following using the definition of logarithm and its properties:

(i) $log_4 16$

(ii) $log_5 125$

(iii) $log_8 1$

(iv) $log_8 8$

(v) $\log 0.1$

2) Assume that $\log_{10} 2 = 0.3010$, find:

(i) $\log_{10} 4$

(ii) $\log_{10} 5$ $\left[\text{Hint: } \log_{10} 5 = \log_{10} \frac{10}{2}\right]$

3) Write each of the following into single logarithm.

i) $log_b z - log_b x - log_b y$

ii) $3\, log_b z - log_b x + 5\, log_b y$

4) If $log_b 2 = 0.69$, $log_b 3 = 1.10$ and $log_b 5 = 1.61$, find the value of the following

i) $log_b \sqrt[3]{2}$

ii) $log_b 2\,7$

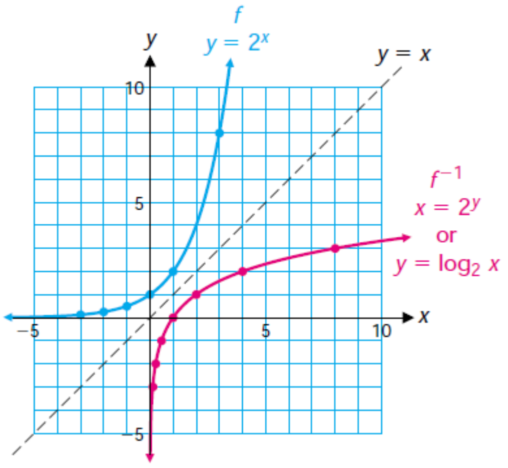
iii) $log_b \frac{5}{3}$

iv) $log_b 1\,5$

7.2 LOGARITHMIC FUNCTION AND GRAPHS

The inverse of exponential function is called logarithmic function.

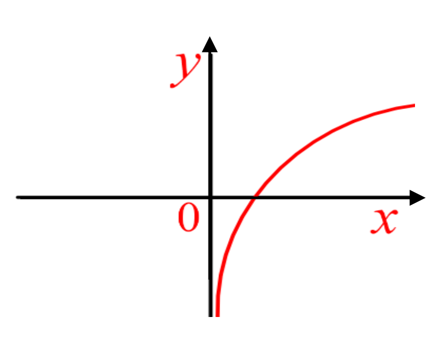
Example, the exponential function $y = 2^x$ has its inverse in the form of $x = 2^y$ in which by logarithm, we write $y = \log_2 x$. Hence, $y = 2^x$ and $y = \log_2 x$ are inverse of each other, Their graphs are symmetric with respect to the line $y = x$



The equation $f(x) = \log_a x$ *or* $y = \log_a x$ where $x > 0$ *and* $a > 0$ *but* $a \neq 1$.
is called a **logarithmic function.**
Domain: $(0,\infty)$, Range: $(-\infty,\infty)$

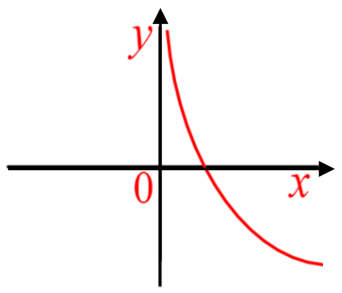
Note: There are two cases in logarithmic functions.

Case (1): $a > 1$



Case(1) If $a > 1$, the graph is an increasing function.

Case (2): $0 < a < 1$



Case(2) If $0 < a < 1$, the graph is a decreasing function.

Remember, if $y = \log_a x$, then $a^y = x$ and conversely, if $a^y = x$, then $y = \log_a x$.

Example 1: Find *x*, *a* or *y* as indicated:

i) Find y **:** $y = \log_4 8$

ii) Find x **:** $\log_3 x = -2$

iii) Find a **:** $\log_a 1000 = 3$

iv) Find x : $\ln x = 2$

Solution:

i)
$y = log_4 8$
$4^y = 8$
$(2^2)^y = 2^3$
$2y = 3$
$y = \frac{3}{2}$

ii)
$\log_3 x = -2$
$x = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

iii)
$\log_a 1000 = 3$
$a^3 = 1000$
$a = (1000)^{\frac{1}{3}}$
$a = 10$

iv) $\ln(x) = 2$ implies that
$\log_e x = 2$, then

$e^2 = x$, so

$x = 7.39$

Or shortly, if $\ln(x) = 2$, then

$x =$ shift $\ln(2)$

$x = 7.39$

Class Activity

1) Find y : $y = \log_{\frac{1}{2}} 8$

2) Find x : $\log_5 x = -2$

3) Find a : $\log_a 8 = 0.5$

7.3 Exponential and Logarithmic Equations

$2^{3x-5} = 4$ is an example of exponential equation and $log(x+3) + log\, x = 1$ is example of logarithmic equation.

Example 1: Solve $2^{3x-2} = 5$ to 2 decimal places.

Solution: $2^{3x-2} = 5$
Taking log on both the sides, we get
$log\, 2^{3x-2} = log\, 5$
$(3x-2)\, log\, 2 = log\, 5$
$(3x-2) = \frac{\log 5}{\log 2}$
$3x = 2 + \frac{\log 5}{\log 2}$
$x = 1.44$

Example 2: Solve $log(x+3) + log\, x = 1$
Solution: $log(x+3) + log\, x = 1$

$$log[x(x+3)] = 1$$
$$x(x+3) = 10^1$$
$$x^2 + 3x - 10 = 0$$
$$(x+5)(x-2) = 0$$
$$x = -5 \text{ or } x = 2$$

Since log of negative value is not defined so $x = 2$

Example 3: Solve

$$log_2[(3x-7)(x-4)] = 3$$

Solution: $log_2[(3x-7)(x-4)] = 3$

$$(3x-7)(x-4) = 2^3$$
$$3x^2 - 19x + 28 = 8$$
$$3x^2 - 19x + 20 = 0$$
$$(3x-4)(x-5) = 0$$

$x = \frac{4}{3}$ or $x = 5$

Example 4:
Solve $ln\ e^{ln\, x} - ln(x-3) = ln\, 2$

Solution: $ln\ e^{ln\, x} - ln(x-3) = ln\, 2$

$$ln\ x - ln(x-3) = ln\, 2$$
$$ln\ \frac{x}{x-3} = ln\, 2$$
$$\frac{x}{x-3} = 2$$
$$x = 2(x-3)$$
$$x = 2x - 6$$
$$x = 6$$

Example 5: Solve $(ln\, x)^2 = ln\, x^2$

Solution: $(ln\, x)^2 = ln\, x^2$

$$(ln\, x)^2 = 2\, ln\, x$$
$$(ln\, x)^2 - 2\, ln\, x = 0$$
$$ln\, x\, (ln\, x - 2) = 0$$
$$ln\, x = 0 \text{ or } ln\, x - 2 = 0$$
$$x = e^0 = 1 \text{ or } x = e^2$$

Class Activity

Solve the following up to 2 decimal places:

1) $2 = 1.002^{4x}$

2) $35^{1-2x} = 7$

3) $log\, x - log\, 5 = log\, 2 - log(\, x - 3)$

4)$ln\, x = ln(\, 2x - 1) - ln(\, x - 2)$

WORKSHEET-7

Section-A

Circle the correct answer in the following questions.

(1) If $log_a 1\,00 = 2$, then 'a' is equal to ……..

(a) 100

(b) 20

(c) 10

(2) If $log_5 x = -3$, then 'x' is equal to ……..

(a) $\frac{1}{125}$

(b) $-\frac{1}{125}$

(c) -15

(3) If $y = log_4 1\,6$, then 'y' is equal to ……..

(a) 4

(b) 2

(c) 12

Section-B

Show your solution step by step in the following questions.

1) Find *x*, *y* or *a* as indicated in the following:

i) $log_5 x = 2$

ii) $log_a 1\,000 = -3$

iii) $y = log_9 2\,7$

2) Solve the following:

i) $log_{10}(5 - x) = 3\, log_{10} 2$

ii) $\log_b(x^2 - 2x - 2) = 2\log_b(x - 2)$

iii) $log(x + 10) = 2 - log\, x$

iv) $\ln x + \ln 4 = 1$

v) $\ln 8 - \ln x = 2$

3) Solve the following:

i) $10^{2x+5} = 43.7$

ii) $e^{1-3x} = 9.62$

4) A certain amount of money *P* (principal) is invested at an annual rate *r* compounded *n* times a year. The amount of money *A* in the account after *t* years, assuming no withdrawals, is given by $A = P\left(1+\frac{r}{n}\right)^{nt}$.

How many years to the nearest year will it take money to double if it is invested at 6% compounded annually (once in year).

Solution:

UNIT 8: LIMITS

8.1 BASIC CONCEPTS OF LIMIT

8.1.1 Functional Notation

In an equation such as $y = 3x^2 + 2x - 5$, y is said to be a function of x and may be written as $y = f(x)$. An equation written in the form $f(x) = 3x^2 + 2x - 5$ is said to have been written in **functional notation**. The value of $f(x)$ when $x = 0$ is denoted by $f(0)$, and the value of $f(x)$ when $x = 2$ is denoted by $f(2)$ and so on. Thus when $f(x) = 3x^2+2x-5$, then

$$f(0) = 3(0)^2 + 2(0) - 5 = -5$$

and $f(2) = 3(2)^2 + 2(2) - 5 = 11$ and so on.

Example 1: If $f(x) = 4x^2 - 3x + 2$ find: $f(0)$ and $f(3) - f(-1)$

$f(x) = 4x^2 - 3x + 2$

then $f(0) = 4(0)^2 - 3(0) + 2 = \mathbf{2}$

$f(3) = 4(3)^2 - 3(3) + 2 = 36 - 9 + 2 = \mathbf{29}$

$f(-1) = 4(-1)^2 - 3(-1) + 2 = 4 + 3 + 2 = \mathbf{9}$

$f(3) - f(-1) = 29 - 9 = \mathbf{20}$

Class Activity 1

1. If $f(x) = 6x^2 - 2x + 1$ find $f(-3)$.

2. If $f(x) = 2x^2 + 5x - 7$ find $f(2) - f(-1)$.

3. If $f(x) = -x^2 + 3x + 6$ find $f(2+a)$ and $\dfrac{f(2+a) - f(2)}{a}$

8.1.2 Definition of Limit of a Function

The tendency of a function when its independent variable approaches some value is called the limit of a function.

Let $f(x)$ be a real valued function, which is defined for all values of x close to $x = c$, with the possible exception of 'c' itself. The function $f(x)$ has limit 'L' when x tends to 'c' from both sides of 'c', **right** and **left,** if $f(x)$ gets closer to L. Symbolically this is written as, $\boldsymbol{\lim_{x \to c} f(x) = L}$.

Figure 1 below provides a visual representation of the mathematical concept of limit.

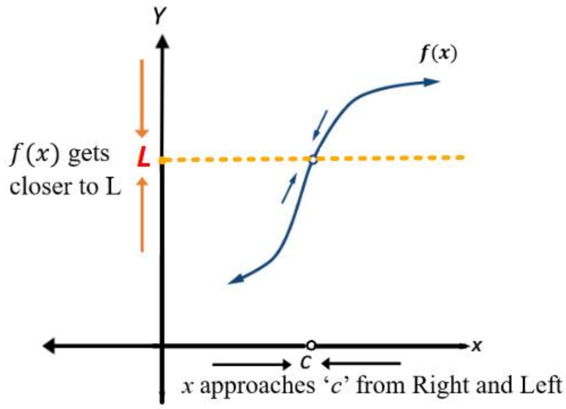


Fig.1

If $\boldsymbol{\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x) = L}$

Then limit of $f(x)$ exists and is equal to L

Thus, $\boldsymbol{\lim_{x \to c} f(x) = L.}$

Example 1:

Consider the graph of the function

$y = f(x) = x + 2$ in fig.1 below.

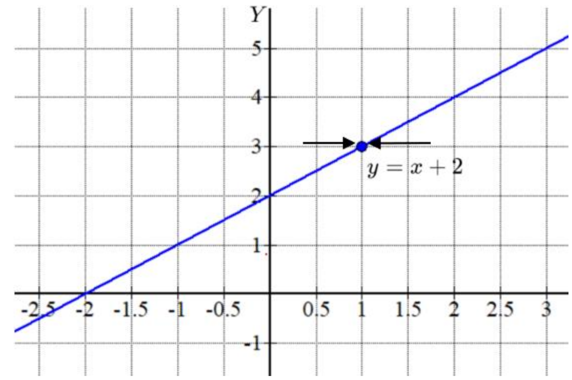


Fig. 1

When x approaches 1 from both sides, **left** and **right**, the function $y = f(x) = x + 2$, approaches 3. Thus, $\lim_{x \to 1}(x + 2) = 3$

Note that without the graph, the same result can be also obtained by evaluating the function for $x = 1$, $ie. f(1) = 1 + 2 = 3$

Example 2:

Consider the graph of the function $f(x) = \frac{1}{x}$ and verify the limit of f(x) as x approaches infinity.

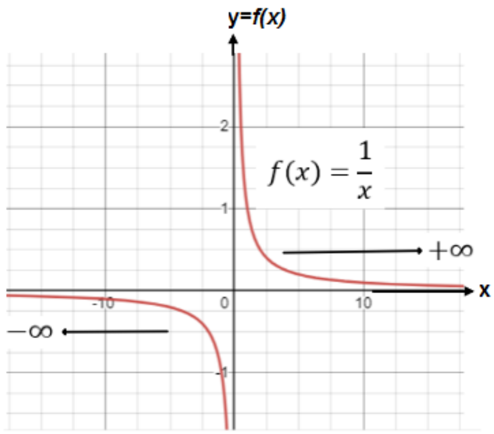


Fig. 2

From the graph in fig.2, as 'x' tends to infinity $'\pm\infty'$, the function $'\frac{1}{x}'$ approaches '0'.

Thus $\lim_{x \to -\infty} f(x) = 0$

and

$$\lim_{x \to +\infty} f(x) = 0$$

Hence $\lim_{x \to \infty} f(x) = 0$

Example 3:

Now consider the graph of the function $f(x) = \frac{1}{x}$ and verify the limit of f(x) as x approaches zero.

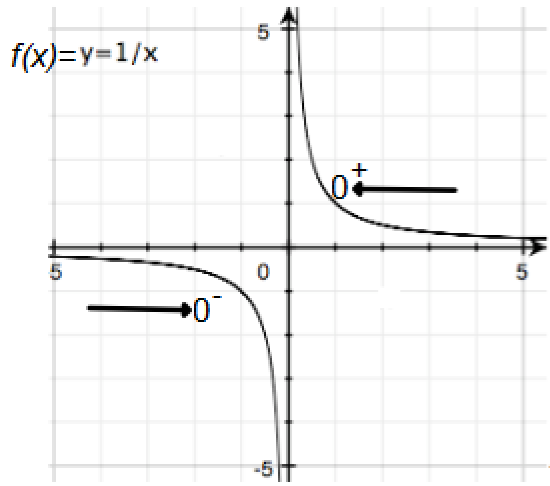


Fig. 3

From the graph in fig.3 above, as 'x' tends to zero from both sides left and right, the graph of the function $y = \frac{1}{x}$ approaches the y-axis but never touches it.

Thus $\lim_{x \to 0^-} f(x) = -\infty$

and

$$\lim_{x \to 0^+} f(x) = +\infty$$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$ we conclude that $\lim_{x \to 0} f(x)$ does not exist.

8.2 METHODS OF FINDING LIMITS

Limits can be found Numerically, Graphically and Algebraically:

Numerical Method

To find the limit *'L'* of a function $f(x)$ as $\boldsymbol{x}$ approaches the number *'c'*, we use some values of $\boldsymbol{x}$ very close to *'c'* and substitute them in the function.

Graphical Method

To find a limit 'L' of a function $f(x)$, sketch the graph of the function and trace the values of *f(x)* as $\boldsymbol{x}$ approaches the number *'c'*.

Algebraic Method

To find a limit 'L' of a function $f(x)$, we use algebraic techniques which usually involve simplification and evaluation of the function.

Example 2: Estimate the limit of the following functions by numerical, graphical and algebraic methods:

i) $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$

Numerical Method
Solution:

$x \to 1^+$	$\dfrac{x^2-1}{x-1}$
1.01	2.01
1.001	2.001
1.0001	2.0001

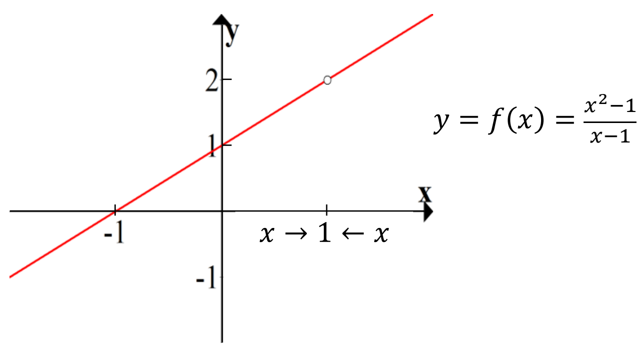
$x \to 1^-$	$\dfrac{x^2-1}{x-1}$
0.9	1.9
0.99	1.99
0.999	1.999

From the table $\lim_{x\to 1}\dfrac{x^2-1}{x-1} = 2$

Note: $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ are called **right hand side limit** and **left hand side limit** respectively or in general **one-sided limits**.
Since the two one sided limits of $f(x)$ are same, we summarize our results by saying that the limit of $f(x)$ as x approaches 1 is 2, written as

$\lim_{x\to 1}\dfrac{x^2-1}{x-1} = 2$ (If the right hand side and left hand side limits are not the same, then the limit does not exist)

Graphical Method



From the graph $\lim_{x\to 1}\dfrac{x^2-1}{x-1} = 2$

Algebraic Method

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}$$
$$= \lim_{x\to 1}(x+1)$$
$$= 2$$

ii) $\lim_{x\to 1} x^3 - 5x$

Solution:

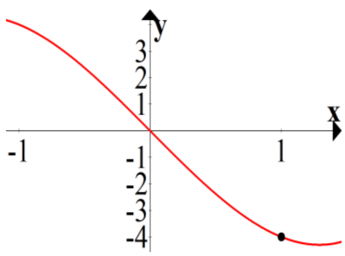
Numerical Method

$x \to 1^+$	$x^3 - 5x$
1.01	-4.02
1.001	-4.002
1.0001	-4.0002

$x \to 1^-$	$x^3 - 5x$
0.9	-3.771
0.99	-3.9797
0.999	-3.9979

From the table $\lim_{x\to 1} x^3 - 5x = -4$

Graphical Method



From the graph $\lim_{x\to 1} x^3 - 5x = -4$

Algebraic Method:

$$\lim_{x\to 1}(x^3 - 5x) = 1^3 - 5 \times 1 = -4$$

iii) $\lim_{x\to 0} e^{2x}$

Solution:

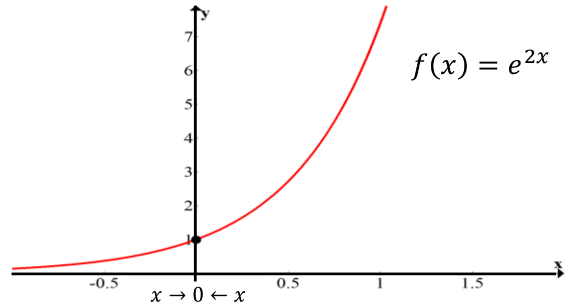
Numerical Method

$x \to 0^+$	e^{2x}
0.01	1.02
0.001	1.002
0.0001	1.0002

$x \to 0^-$	e^{2x}
-0.01	0.98
-0.001	0.998
-0.0001	0.9998

From the table $\lim_{x\to 0} e^{2x} = 1$

Graphical Method

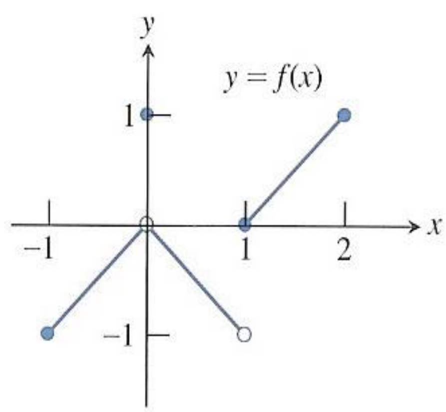


From the graph $\lim_{x\to 0} e^{2x} = 1$

Algebraic Method: $\lim_{x\to 0} e^{2x} = e^0 = 1$

<u>Class Activity 2</u>

1. Which of the following statements about the function $y = f(x)$ graphed below are true and which are false?



a) $\lim_{x\to 0} f(x)$ exists

b) $\lim_{x\to 0} f(x) = 0$

c) $\lim_{x\to 0} f(x) = 1$

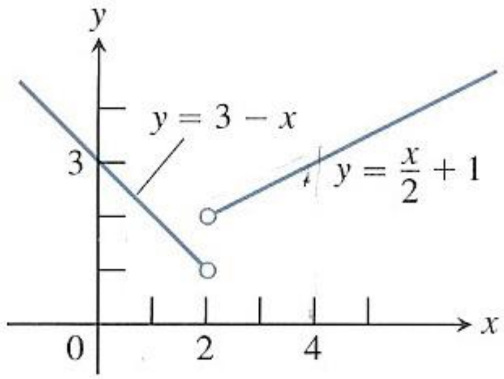
d) $\lim_{x\to 1} f(x) = 1$

e) $\lim_{x\to 1} f(x) = 0$

2. In the figure below $y = f(x)$

a) Find $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$

b) Does $\lim_{x \to 2} f(x)$ exist? Give a reason.



3. Determine the following limits algebraically if they exist:

a) $\lim_{x \to -3} \left(\frac{x^2-9}{x+3}\right)$

b) $\lim_{y \to 1} \left(\frac{y+1}{y-1}\right)$

c) $\lim_{x \to 3} \left(3x + \frac{1}{3x}\right)$

4. Determine the $\lim_{x \to 2} h(x)$ when h is defined as follows:

$$h(x) = \begin{cases} \frac{3x}{2}, & if\ x < 2 \\ 3x + 4, & if\ x \geq 2 \end{cases}$$

8.3 LIMITS AT INFINITY

For each of the following functions f, evaluate $\lim_{x\to\infty} f(x)$.

a) $f(x) = 5x^3$

Solution: $\lim_{x\to\infty} 5x^3 = \infty$

b) $f(x) = 1 + \frac{2}{x}$

Solution: $\lim_{x\to\infty}\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} 1 + \lim_{x\to\infty}\left(\frac{2}{x}\right)$

$$L = 1 + 0 = 1$$

(since $\lim_{x\to\infty} \frac{2}{x} = 0$)

c) $f(x) = \frac{x-1}{2x+3}$

Solution: $\lim_{x\to\infty}\left(\frac{x-1}{2x+3}\right) = \lim_{x\to\infty}\left(\frac{\frac{x}{x}-\frac{1}{x}}{\frac{2x}{x}+\frac{3}{x}}\right) =$

$$\lim_{x\to\infty}\left(\frac{1-\frac{1}{x}}{2+\frac{3}{x}}\right) = \frac{1}{2}$$

(Note that $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty} \frac{3}{x} = 0$)

d) $f(x) = \frac{3x+2}{x^2+x}$

Solution: $\lim_{x\to\infty}\left(\frac{3x+2}{x^2+x}\right) = \lim_{x\to\infty}\left(\frac{\frac{3x}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{x}{x^2}}\right)$

$$= \lim_{x\to\infty}\left(\frac{\frac{3}{x}+\frac{2}{x^2}}{1+\frac{1}{x}}\right) = \frac{0}{1} = 0$$

Class Activity 3

Determine the following limits if they exist.

1) $\lim_{x\to\infty}\left(7 - \frac{5}{3x^2}\right)$

2) $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)$

3) $\lim_{x\to\infty}(9x^2 + 2x + 1)$

4) $\lim_{x\to\infty}\left(\frac{5x-2}{3x+7}\right)$

5) $\lim_{x\to\infty}\left(\frac{9x+5}{x^2+2}\right)$

8.4 CONTINUITY OF A FUNCTION

A function $f(x)$ is continuous at $x=c$ if and only if it meets the three conditions:

1. $f(c)$ exists
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

The following procedure can be used to analyze the continuity of a function at a given point "c".

Step 1: Check to see if ***f(c)*** is defined, if ***f(c)*** is not defined, then the function is not continuous at point "c" and we need go no further. If ***f(c)*** is defined, continue to step 2.

Step 2: Evaluate $\lim_{x\to c} f(x)$ by computing $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$, if $\lim_{x\to c} f(x)$ does not exist , then the function is not continuous at point "c". If $\lim_{x\to c} f(x)$ exists, continue to step 3.

Step 3: If $\lim_{x\to c} f(x) \neq f(c)$, then the function is not continuous at point "c".

If $\lim_{x\to c} f(x) = f(c)$, then the function is continuous at point "c"

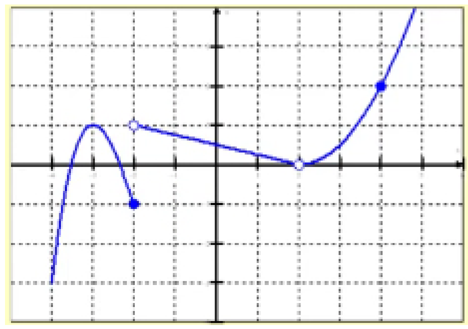
Note: $f(x)$ is continuous at $x=c$ if

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

Example 1:

Determine if the following function is continuous at:

a) $x = -2$
b) $x = 2$
c) $x = 4$



Solution:

a) $f(-2) = -1,\ ie. f(-2)\ exists$

$\lim_{x\to -2^-} f(x) = -1$

$\lim_{x\to -2^+} f(x) = 1$

$\lim_{x\to -2} f(x)$ does not exist since

$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

Therefore the function is not continuous at $x = -2$

b) $f(2)$ does not exist, therefore the function is not continuous at $x = 2$

c) $f(4) = 2$ (exists)

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = 2$

$\lim_{x\to 4} f(x) = 2$(exists)

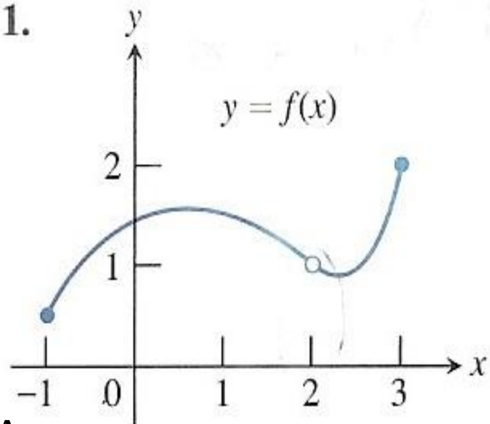
$\lim_{x\to 4} f(x) = 2 = f(4)$

Therefore the function is continuous at $x = 4$

Class Activity 4

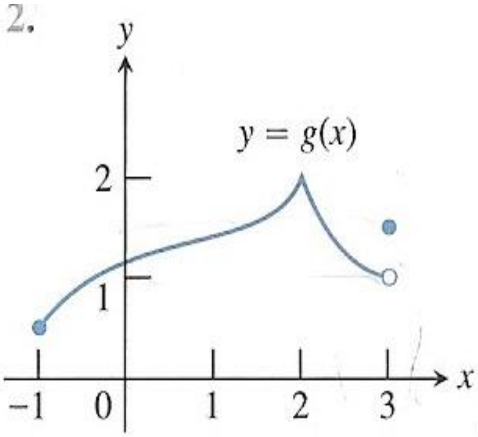
In figures 1 - 4 state whether the function graphed is continuous on [-1,3] or not. If not, give a reason.

1.



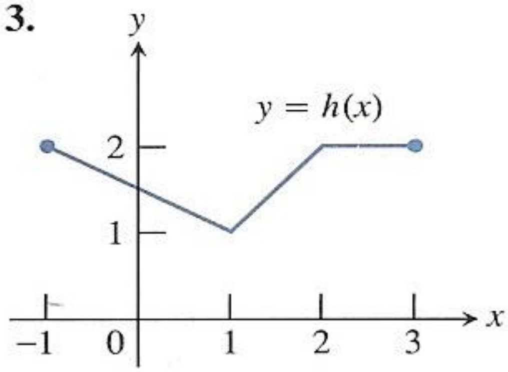
Ans:

2.



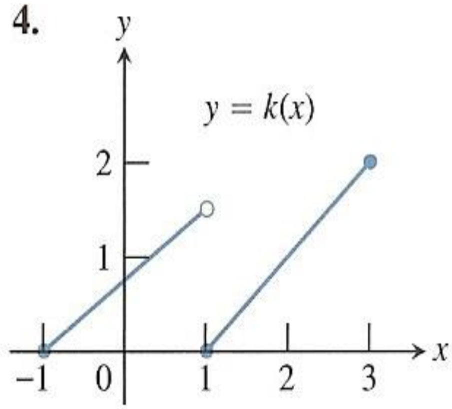
Ans:

3.



Ans:

4.



Ans:

5. Determine whether the function

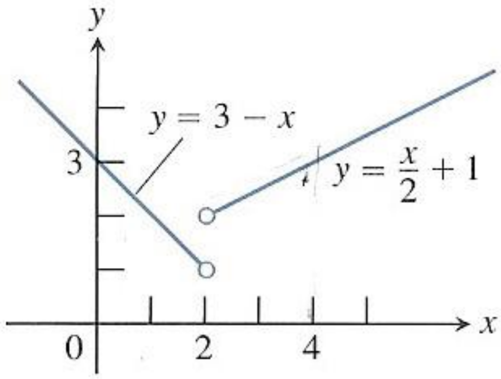
$$f(x) = \begin{cases} -x^2 \ if \ x \le 4 \\ 5x - 9 \ if \ x > 4 \end{cases}$$

is continuous at $x = 4$

Solution:

WORKSHEET 8

1.



From the above graph,

a) Find $\lim_{x\to 2^+} f(x)$

and $\lim_{x\to 2^-} f(x)$

b) Does $\lim_{x\to 2} f(x)$ exist?

Give a reason.

c) Find $\lim_{x\to 4^+} f(x)$

and $\lim_{x\to 4^-} f(x)$

d) Does $\lim_{x\to 4} f(x)$ exist?

Give a reason.

2. Evaluate the following limits

a) $\lim_{x\to 6} \frac{x}{3} =$

b) $\lim_{x\to 2} \frac{x^2-4}{x-2} =$

c) $\lim_{x\to \frac{1}{2}} \frac{2x-1}{4x^2-1} =$

d) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} =$

e) $\lim_{x\to 1} \left(\frac{x+1}{x-1}\right)$

f) $\lim_{y\to 2} \left(2y + \frac{1}{2y}\right)$

Evaluate the following limits

3. $\lim_{x \to -3} \frac{x^2-9}{x+3} =$

4. $\lim_{x \to 4} \frac{x^2+\mathrm{x}-20}{x-4} =$

5. $\lim_{x \to 1.5} \frac{2x-3}{6x^2-13x+6} =$

6. $\lim_{x \to \frac{1}{9}} \frac{9x-1}{3\sqrt{x}-1} =$

7. Determine whether the function

$$f(x) = \begin{cases} -x^2 \ if \ x \leq 3 \\ 4x - 8 \ if \ x > 3 \end{cases}$$

is continuous at $x = 3$

8. Determine whether the function

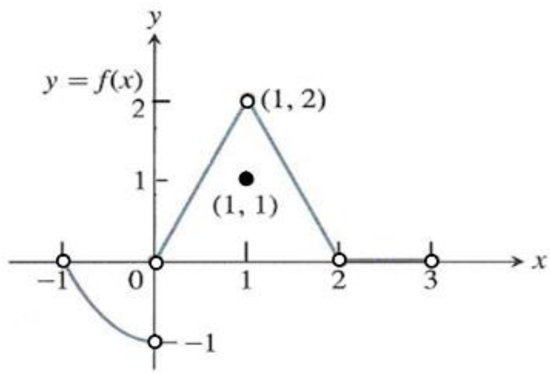
$$f(x) = \begin{cases} (x-2)^2 \ if \ x < 1 \\ \frac{1}{x-1} \ if \ x \geq 1 \end{cases}$$

has a limit when $x = 1$

9. What is $\lim_{x \to \infty} \frac{1}{x-1}$?

10. What is $\lim_{x \to \infty} \frac{x}{x^2+1}$?

Consider the following graph and answer the questions from 11-13:



11.

a) Does f (1) exists?

b) Does $\lim_{x \to 1} f(x)$ exist?

c) Does $\lim_{x \to 1} f(x) = f(1)$?

d) Is f continuous at $x = 1$?

12.

a) Is f defined at $x = 2$?

b) Is f continuous at $x = 2$?

c) What value should be assigned to $f(2)$ to make the function continuous at $x = 2$?

13. Determine the intervals in which the function is continuous.

14. For the function $f(x)$ defined below, determine the value of b so that $\lim_{x \to 5} f(x)$ exists.

$$f(x) = \begin{cases} 2x - 3 \; if \; x < 5 \\ \frac{2}{3}x + b \; if \; x \geq 5 \end{cases}$$

UNIT 9: DIFFERENTIATION

9.1 THE GRADIENT (SLOPE) OF A CURVE

(a) A **tangent line** is a straight line that touches a function at only one point (Fig.3.1). The tangent line represents the **instantaneous rate of change** of the function at that one point. If a tangent is drawn at a point *P* on a curve, then the gradient of this tangent is said to be the **gradient of the curve** at *P*. In Fig. 3.1, the gradient of the curve at *P* is equal to the gradient of the tangent PQ

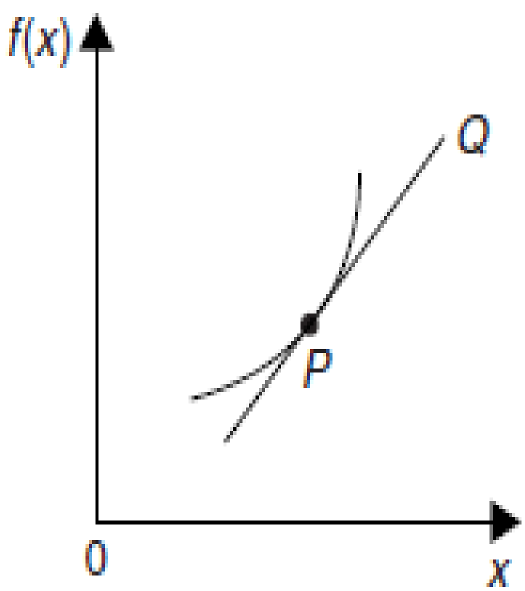


Fig. 3.1

(b) For the curve shown in Fig. 3.2, let the points *A* and *B* have co-ordinates (x_1, y_1) and (x_2, y_2), respectively. In functional notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$ as shown.

The gradient of the chord *AB (secant line)* (straight line joining A and B)

$$= \frac{BC}{AC} = \frac{BD - CD}{ED} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

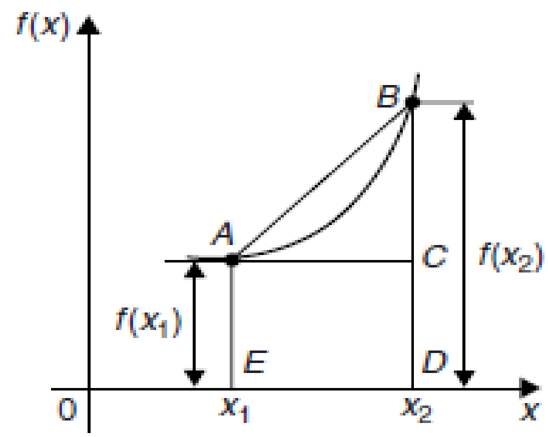


Fig. 3.2

(c) For the curve $f(x) = x^2$ shown in Fig. 3.3

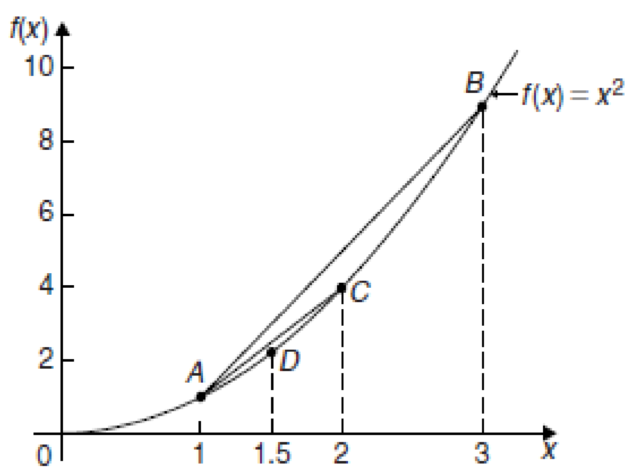


Fig. 3.3

AB = $\dfrac{f(3)-f(1)}{3-1} = \dfrac{9-1}{2} = 4$

(ii) The gradient of the chord

AC = $\dfrac{f(2)-f(1)}{2-1} = \dfrac{4-1}{1} = 3$

(iii) The gradient of the chord

AD = $\dfrac{f(1.5)-f(1)}{1.5-1} = \dfrac{2.25-1}{0.5} = 2.5$

(iv) If E is the point on the curve

(1.1, f(1.1)) then the gradient of the chord

AE = $\dfrac{f(1.1)-f(1)}{1.1-1} = \dfrac{1.21-1}{0.1} = 2.1$

(v) If F is the point on the curve

(1.01, f(1.01)) then the gradient of the chord

AF = $\dfrac{f(1.01)-f(1)}{1.01-1} = \dfrac{1.0201-1}{0.01} = 2.01$

Thus, as point *B* moves closer and closer to point *A*, the gradient of the chord approaches nearer and nearer to value 2. This is called the **limiting value** of the gradient of the chord *AB* and when *B* coincides with *A* the chord becomes the tangent to the curve.

Therefore, the limit of the gradient(slope) of the chord AB = value of the gradient of the tangent line at point A, which is equal to 2.

9.2 DIFFERENTIATION FROM THE FIRST PRINCIPLES

Introduction

Calculus is the mathematical study of continuous change, in the same way that Geometry is the study of shape, and Algebra is the study of generalisation of arithmetic operations. Two mathematicians, Isaac Newton of England and Gottfried Wilhelm Leibniz of Germany share credit for having independently developed the calculus in the 17th century. There are two branches of calculus:

1. **Differential calculus**, which deals with finding the rate of change of a quantity and,
2. **Integral calculus**, which deals with finding the quantity when the rate is known.

In this section, we shall be limited to the study of Differential calculus only.

(i) In **Fig. 4.1**, *A* and *B* are two points very close together on a curve, δx (delta *x)* and δy (delta *y)* representing small increments in the *x* and *y* directions, respectively.

The gradient of the chord

$$AB = \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

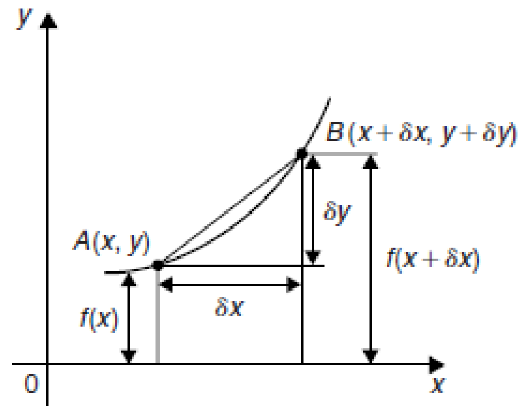


Fig. 4.1

As point B moves closer to point A, δx approaches zero and $\delta y/\delta x$ approaches a limiting value and the gradient of the chord approaches the gradient of the tangent at *A*.

(ii) When determining the gradient of a tangent to a curve there are two notations used. The gradient of the curve at *A* in Fig. 4.1 can either be written as:

$$\underset{\delta x \to 0}{\text{limit}} \frac{\delta y}{\delta x} \quad \text{or} \quad \underset{\delta x \to 0}{\text{limit}} \left[\frac{f(x+\delta x) - f(x)}{\delta x} \right]$$

In **Leibniz notation**, $\frac{dy}{dx} = \underset{\delta x \to 0}{\text{limit}} \frac{\delta y}{\delta x}$

In **functional notation**,

$$f'(x) = \underset{\delta x \to 0}{\text{limit}} \left[\frac{f(x+\delta x) - f(x)}{\delta x} \right]$$

(iii) $\frac{dy}{dx}$ is the same as $f'(x)$ or y' and is called the **differential coefficient** or the **derivative**. The process of finding the differential coefficient is called **differentiation**.

Summarising, the differential coefficient,

$$\frac{dy}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x + \delta x) - f(x)}{\delta x}\right]$$

Example 1: Differentiate from first principles $f(x) = x^2$ and determine the value of the gradient of the curve at $x = 2$

Solution: To 'differentiate from first principles' means 'to find $f'(x)$' by using the expression $f'(x) = \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$

Here $f(x) = x^2$

$$f'(x) = \lim_{\delta x \to 0}\left[\frac{(x+\delta x)^2 - x^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{x^2 + 2x\delta x + \delta x^2 - x^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{2x\delta x + \delta x^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}[2x + \delta x]$$

$$= 2x + 0 = 2x$$

Thus $f'(x) = 2\boldsymbol{x}$, i.e. the differential coefficient of x^2 is $2x$. At $x = 2$, the gradient of the curve,

$f(x) = 2(2) = \mathbf{4}$

Note: '**differential coefficient'**, '**finding the derivative', 'finding the gradient'** all have same meaning.

<u>Class Activity 1</u>

1. Using first principles, find the differential coefficient of $y = 4x$

2. Find the derivative of $y = 8$, using first principles.

3. Differentiate from the first principles $f(x) = 2x^3$

9.3 METHODS OF DIFFERENTIATION

Differentiation from first principles can be a lengthy process and it would not be convenient to go through this procedure every time we want to differentiate a function. Instead, we better use rules of differentiation which were derived from the definition of derivative or from the first principles.

9.3.1 General Rules of Differentiation

1. $\frac{d(c)}{dx} = 0$ where **c** is any constant.

   Example: If $y = 5$**, then**

   $$\frac{dy}{dx} = \frac{d(5)}{dx} = 0$$

2. $\frac{d}{dx}[a.f(x))] = a.\frac{d\,f(x)}{dx}$

   Example: If $y = 7x$, then

   $$\frac{dy}{dx} = \frac{d(7x)}{dx} = \frac{7\,d(x)}{dx} = 7$$

3. **The Power Form**:

   $$\frac{d(x^n)}{dx} = nx^{n-1}$$

   Example: If $y = x^3$, then

   $$\frac{dy}{dx} = \frac{d(x^3)}{dx} = 3\,x^{3-1} = 3\,x^2$$

4. **Derivative of Sum or Difference of Two Functions**

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Example: If $y = x^2 - x + 4$, then

$$\frac{dy}{dx} = \frac{d(x^2)}{dx} - \frac{d(x)}{dx} + \frac{d(4)}{dx} = 2x - 1$$

5. **Derivative of a Product**

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x).f'(x)$$

or

$$\frac{d}{dx}[u.v)] = u.\frac{dv}{dx} + v.\frac{du}{dx} = uv' + u'v$$

where u and v are two different functions of x.

Example: If $y = 2x\sqrt{x+2}$, then

$$\frac{dy}{dx} = 2x\,.\,\frac{d}{dx}\left(\sqrt{x+2}\right) + \sqrt{x+2}\,.\frac{d}{dx}(2x)$$

$$= 2x\left[\frac{1}{2}(x+2)^{-1/2}(1)\right] + \sqrt{x+2}\;(2)$$

$$= \frac{x}{\sqrt{x+2}} + 2\sqrt{x+2}$$

6. **Derivative of a Quotient**

   When $y = \frac{u}{v}$ where u and v are both functions of x, then

   $$y' = \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{u'v - uv'}{v^2}$$

   alternatively, if $y = \frac{f(x)}{g(x)}$,

   then $\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Example: If $y = \frac{x^2-1}{3x}$, find $\frac{dy}{dx}$.

Solution: $\frac{x^2-1}{3x}$, is a quotient.

Let $u = x^2 - 1$ and $v = 3x$

$$y' = \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{3x\,\frac{d(x^2-1)}{dx} - (x^2-1)\frac{d(3x)}{dx}}{(3x)^2}$$

$$= \frac{3x(2x) - (x^2-1)(3)}{9x^2}$$

$$= \frac{6x^2 - 3x^2 + 3}{9x^2}$$

$$= \frac{3x^2+3}{9x^2} = \frac{3(x^2+1)}{3(3x^2)}$$

$$\therefore \frac{dy}{dx} = \frac{x^2+1}{3x^2}$$

Class Activity 2

Differentiate the following functions:

1. $y = 3x^2 - 2x + 3$

2. $y = \frac{4}{3x^2}$

3. $y = (4x^2)\sqrt[3]{x+1}$

4. $y = \frac{2x^2 + 3x - 2}{\sqrt{x}}$

9.3.2 Differentiation of function of a function (composite functions)

If $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is known as the function of a function rule (or sometimes the C**hain Rule**). It is often easier to make a substitution before differentiating.

Example 1. Find the derivative of

$y = (3x-1)^5$

Solution: if $y = (3x-1)^5$ then, by making the substitution $u = (3x-1)$, we get $y = u^5$, which is of the 'standard' form.

Hence, $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = 3$

Then, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (5u^4)(3) = 15u^4$

Rewriting *u* as $(3x-1)$ gives:

$$\frac{dy}{dx} = 15(3x-1)^4$$

Since *y* is a function of *u*, and *u* is a function of *x*, then *y* is a function of a function of *x*.

Example 2. Find the derivative of

$$y = (4t^3 - 3t)^6$$

Solution: Let $u = 4t^3 - 3t$, then $y = u^6$

Hence, $\dfrac{dy}{du} = 6u^5$ and $\dfrac{du}{dt} = 12t^2 - 3$

Using the function of a function rule,

$$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = (6u^5)(12t^2 - 3)$$

Rewriting *u* as $(4t^3 - 3t)$ gives:

$$\frac{dy}{dt} = 6(4t^3 - 3t)^5(12t^2 - 3)$$

$$= 18(4t^3 - 3t)^5(4t^2 - 1)$$

Example 3. Determine the differential coefficient of: $y = \sqrt{3x^2 + 4x - 1}$

Solution: $y = \sqrt{3x^2 + 4x - 1}$

$$or \quad y = (3x^2 + 4x - 1)^{1/2}$$

Let $u = 3x^2 + 4x - 1$ then $y = u^{1/2}$

Hence

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \text{ and } \frac{du}{dx} = 6x + 4$$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{1}{2\sqrt{u}}\right)(6x + 4)$$

$$= \frac{3x + 2}{\sqrt{u}} = \frac{3x + 2}{\sqrt{3x^2 + 4x - 1}}$$

The General Power Form:

From the **Chain Rule,** the **General Power form** can now be written as:

$$\boldsymbol{\frac{d(u^n)}{dx} = nu^{n-1}\frac{d\,(u)}{dx}} \text{ where } u = f(x).$$

Example: If $y = (2x^3 - 5x)^5$, find $\frac{dy}{dx}$.

Solution: Let $u = 2x^3 - 5x$, and $n = 5$

$$\frac{du}{dx} = 6x^2 - 5$$

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{d\,(u)}{dx}$$

$$\frac{dy}{dx} = 5(2x^3 - 5x)^4(6x^2 - 5)$$

Class Activity 3

Find the derivative of the following functions**:**

1. $y = (2x - 1)^6$

2. $y = \frac{1}{(x^3 - 2x + 5)^5}$

3. $y = \sqrt[3]{6x - 2}$

9.3.3 Derivatives of Trigonometric Functions

1. $\frac{d}{dx}(\boldsymbol{sin\, u}) = \boldsymbol{cos\, u} \, . \frac{d\,(u)}{dx}$ where $u = f(x)$.

2. $\frac{d}{dx}(\boldsymbol{cos\, u}) = -\boldsymbol{sin\, u} \, . \frac{d\,(u)}{dx}$

3. $\frac{d}{dx}(\boldsymbol{tan\, u}) = \boldsymbol{sec^2\, u} \, . \frac{d\,(u)}{dx}$

Extra Rules

4. $\frac{d}{dx}(\boldsymbol{cot\, u}) = -\boldsymbol{csc^2\, u} \, . \frac{d\,(u)}{dx}$

5. $\frac{d}{dx}(\boldsymbol{sec\, u}) = \boldsymbol{sec\, u \, . tan\, u} \, \frac{d\,(u)}{dx}$

6. $\frac{d}{dx}(\boldsymbol{csc\, u}) = -\boldsymbol{csc\, u \, . cot\, u} \, \frac{d\,(u)}{dx}$

Example1: Find the derivative of $\boldsymbol{y = \sin 3x}$.

Solution: Use $\frac{d}{dx}(\boldsymbol{sin\, u}) = \boldsymbol{cos\, u} \, . \frac{d\,(u)}{dx}$

$u = 3x$

$\frac{dy}{dx} = \cos 3x. \ \frac{d}{dx}(3x)$

$\frac{dy}{dx} = 3 \ \cos 3x.$

Example2: Find $\frac{dy}{dx}$ of $\boldsymbol{y = \tan 2x}$.

Solution: Use $\frac{d}{dx}(\boldsymbol{tan\, u}) = \boldsymbol{sec^2\, u} \, . \frac{d\,(u)}{dx}$

$\frac{dy}{dx} = \boldsymbol{sec^2}\, 2x. \ \frac{d}{dx}(2x)$

$\frac{dy}{dx} = 2\, \boldsymbol{sec^2}\, 2x$

9.3.4 Derivatives of Exponential Functions

Let ***a*** be any real number but not zero and

$u = f(x)$

1. $\frac{d}{dx}(\boldsymbol{a^u}) = \boldsymbol{a^u\, lna} . \frac{d\,(u)}{dx}$

2. $\frac{d}{dx}(\boldsymbol{e^u}) = \boldsymbol{e^u} \, . \frac{d\,(u)}{dx}$

Example1. Given $y = \boldsymbol{4^{cos\, x}}$, find $\frac{dy}{dx}$.

Solution: Use $\frac{d}{dx}(\boldsymbol{a^u}) = \boldsymbol{a^u\, lna} . \frac{d\,(u)}{dx}$

$\frac{dy}{dx} = 4^{cos\, x}(\ln 4)\, \frac{d}{dx}(cosx)$

$\frac{dy}{dx} = 4^{cos\, x}(\ln 4)\,(-sin\, x)$

$\frac{dy}{dx} = -(\ln 4)\, 4^{cos\, x} sin\, x$

Example2. Find $\frac{dy}{dx}$ of $y = \boldsymbol{e^{2x}}$.

Solution:

$\frac{dy}{dx} = \boldsymbol{e^{2x}}\, \frac{d}{dx}(2x)$

$\frac{dy}{dx} = \boldsymbol{2e^{2x}}$

9.3.5 Derivatives of Logarithmic Functions

Let ***a*** be any real number but not zero and

$u = f(x)$

1. $\frac{d}{dx}(\boldsymbol{\log_a u}) = \frac{\boldsymbol{1}}{\boldsymbol{u\, lna}} \, \frac{d\,(u)}{dx}$

2. $\frac{d}{dx}(\boldsymbol{ln\, u}) = \frac{\boldsymbol{1}}{\boldsymbol{u}} \, \frac{d\,(u)}{dx}$

Example1. Find the derivative of

$$y = \boldsymbol{\log_2}(\sqrt{3x + 4})$$

Solution:

$y = \frac{1}{2}\log_2(\, 3x + 4)$ by properties of logarithm.

$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\, \{\log_2(\, 3x + 4)\}$

$\frac{dy}{dx} = \frac{1}{2(3x+4)\, ln2} \ \frac{d\,(3x+4)}{dx}$

$\frac{dy}{dx} = \frac{3}{2ln2\ (3x+4)}$

Example2. Find the derivative of

$\boldsymbol{y = \ln(\sin x)}$ **.**

Solution: Use $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{d(u)}{dx}$

$$\frac{dy}{dx} = \frac{1}{\sin x}\frac{d}{dx}(\sin x)$$

$$\frac{dy}{dx} = \frac{1}{\sin x}\ cosx\frac{d(x)}{dx} = \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = \cot x$$

Class Activity 4

Find the differential coefficient or derivative of the following functions:

1 $y = 2sin3x - 4\,cos2x$

2. $y = 3x^2 sin\,2x$

3. $y = \ln(\cos 3x)$

4. $y = \sqrt{x^3}\,ln\,3x$

5. $y = \frac{1-\sqrt{x}}{e^x}$

6. $y = \log_3(5x-3)^4$

7. $y = \frac{2\,cos\,3x}{x^3}$

8. $y = \boldsymbol{e^{tanx}}$

9.3.6 Interpretation of Derivative

1) The Derivative represents the gradient (slope) of the tangent line to the curve at a specific point on the curve.

Example: Find the gradient of the curve $y = x^3 + 4x^2 + x - 2$ at the point (1, 2).

Solution: we have $y = x^3 + 4x^2 + x - 2$ so the gradient =

$$\frac{dy}{dx} = 3x^2 + 8x + 1$$

and at the point (1, 2), we have $x = 1$

Thus, the slope or gradient at the point (1, 2) = $3(1)^2 + 8(1) + 1 = 12$

Example: Determine the co-ordinates of the point on the curve $y = x^2 - 5x - 7$, where the gradient is -1.

Solution: When $y = x^2 - 5x - 7$ then

$$\text{gradient } = \frac{dy}{dx} = 2x - 5$$

Since gradient is -1 so $2x - 5 = -1$ which gives $x = 2$

When $x = 2$ then $y = (2)^2 - 5(2) - 7$

$$= -13$$

Therefore, the gradient is -1 at the point $(2, -13)$

Class Activity 5

1. Find the gradient of the curve: $y = 2t^4 + 3t^3 - t + 4$ at the point (0, 4).

2. Find the differential coefficient of $y = 4x^2 + 5x - 3$ and determine the gradient of the curve at $x = -3$

3. Find the co-ordinates of the point on the graph $y = 5x^2 - 3x + 1$ where the gradient is 2.

9.4 APPLICATIONS OF DERIVATIVES

In this section, we look at the application of derivative by focusing on the interpretation of derivative as the rate of change of a function.

Example: Find the rate of change of y with respect to x given: $y = 3\sqrt{x}\, ln\, 2x$

Solution: The rate of change of y with respect to x is given by $\frac{dy}{dx}$

$y = 3\sqrt{x}\, ln\, 2\, x = 3x^{1/2}\, ln\, 2\, x$, which is a product.

Let $u = 3x^{1/2}$ and $ln\, 2\, x$

Then the product rule:

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'$$

gives:

$$\frac{dy}{dx} = \left(3x^{1/2}\right)\left(\frac{1}{x}\right) + (ln2x)\left(3\frac{1}{2}x^{1/2-1}\right)$$

$$\frac{dy}{dx} = 3x^{1/2-1} + (ln2x)\left(\frac{3}{2}x^{-1/2}\right)$$

$$\frac{dy}{dx} = 3x^{-1/2} + (ln2x)\left(\frac{3}{2}x^{-1/2}\right)$$

$$\frac{dy}{dx} = 3x^{-1/2}\left(1 + \frac{1}{2}ln2x\right)$$

$$i.e. \quad \frac{dy}{dx} = \frac{3}{\sqrt{x}}\left(1 + \frac{1}{2}ln\, 2\, x\right)$$

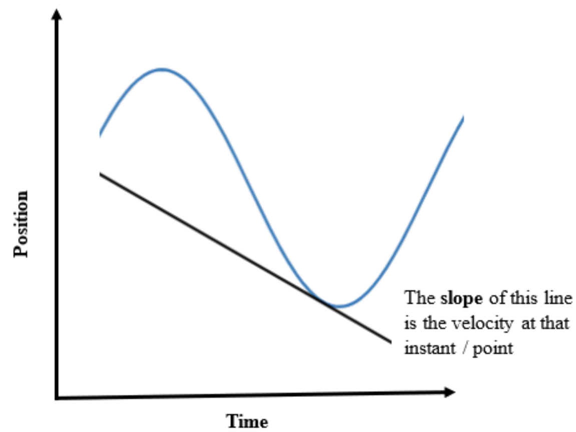
In Physics, derivatives are applied to calculate Velocity and Acceleration. In linear motion, velocity is the rate of change of position and acceleration is the rate of change of velocity.

Definition:

Let $s(t)$ be a function giving the position of an object at time t:

1. The velocity of the object at time t is given by $v(t) = \frac{ds}{dt} = s'(t)$

We find the **velocity** at any instant or point by looking at the **slope** of the **tangent line** on a position curve



Example: The distance s moved by a body in t seconds is given by $s = t^3 - 3t^2$. What is the velocity of the body after 5 seconds?

Solution: The velocity of the body at time t is given by $v(t) = \frac{ds}{dt}$

Since $s = t^3 - 3t^2$, then

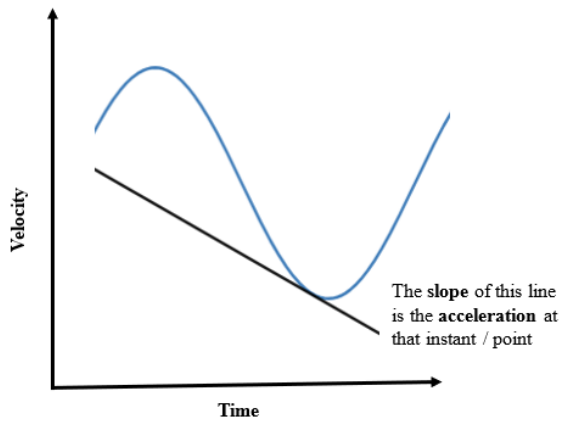
$$v(t) = \frac{ds}{dt} = 3t^2 - 6t$$

When $t = 5$, $\frac{ds}{dt} = 3 \times 5^2 - 6 \times 5 = 45$

Therefore the velocity of the body after 5 seconds is $45m/s$

2. The acceleration of the object at time t is given by $a(t) = \frac{dv}{dt} = v'(t)$

We find the **acceleration** at any instant or point by looking at the **slope** of the **tangent line** on a **velocity** curve.



Example:

The distance s moved by a body in t seconds is given by $s = t^3 - 3t^2$. What is the acceleration of the body after 3 seconds?

Solution: The acceleration of the body at time t is given by $a(t) = \frac{dv}{dt} = v'(t)$

Since $v(t) = \frac{ds}{dt} = 3t^2 - 6t$, then

$a(t) = \frac{dv}{dt} = 6t - 6$

When $t = 3$, $\frac{dv}{dt} = 6 \times 3 - 6 = 12$

Therefore acceleration of the body after 3 seconds $12m/s^2$

Class Activity 6

1. An alternating current is given by

$i = 5sin100t$ amperes, where t is the time in seconds. Determine the rate of change of current i when $t = 0.01$ seconds.

(Round off answer to 1 decimal place)

Solution:

2. Determine the rate of change of voltage, given $v = 5t\, sin\, 2t\ volts$, when $t = 0.2$

(Round off answer to 3 significant figures)

Solution:

3. A particle moves *s* metres in *t* seconds according to the relationship $s = t^3 - 7t - 3$

a) find the velocity of the particle after 5 seconds

b) find the acceleration of the particle after 3 seconds

4. The distance *s* moved by a body in time *t* is given by the function $s = 40t - 5t^2$. Calculate the time taken for the body to come to rest.

WORKSHEET 9

In problems 1 to 5, determine the differential coefficient with respect to the variable.

1. $y = 2x^3 - 5x + 6$

2. $y = 5x\sqrt{x+3}$

3. $y = x - \frac{1}{x^2}$

4. $y = \frac{3x^2 + 5x - 2}{\sqrt{x}}$

5. $y = y = e^{cosx}$

6. Determine the gradient of the curve $y = -2x^3 + 4x + 7$ at $x = -1.5$

7. Find the co-ordinates of the point on the graph $y = 5x^2 - 3x + 1$ where the gradient is 2.

8. Find the gradient of the curve $y = 2cos\frac{1}{2}x$ at $x = \frac{\pi}{2}$

9. Determine the gradient of the curve $y = 3sin2x$ at $x = \frac{\pi}{3}$

10. Differentiate with respect to *x*

$2e^x ln2x$

11. Determine the derivative of

$y = \ln(sin\,2x)$

12. Differentiate $y = (x^2 + 1)cosx$

13. Differentiate $y = \frac{x^2+1}{x+1}$

14. If $y = \frac{6\,cos\,5x}{x^5}$, determine $\frac{dy}{dx}$

15. Differentiate $y = (x^3 - x)^{-3}$

16. Determine $\frac{dy}{dx}$ for the function:

$y = \sqrt[5]{2 + 3x^2 - x^3}$

17. Determine the rate of change of voltage, given $v = 5t\,sin\,2t$ *volts*, when $t = 0.2$

18. Power P and voltage V of a lamp are related by $P = aV^b$ where a and b are constants. Find an expression for the rate of change of power with voltage.

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